Five Trigonometry Identities problems

1. If $A + B + C = 180^o$,
prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

2. Prove that $1 - \cos 2A + \cos 4A - \cos 6A = 4 \sin A \cos 2A \sin 3A$

3. Prove that $\tan 4A (\sin 2A + \sin 10A) = \cos 2A - \cos 10 A$

4. Prove that $\sin A (\sin 3A + \sin 5A) = \cos A (\cos 3A - \cos 5A)$

5. Prove that $\tan A + \tan (A + 120^o) + \tan (A + 240^o) = 3 \tan 3A$

Solutions

1. $A + B + C = 180^o \Rightarrow A + B = 180^o - C$

$\tan(A + B) = \tan (180° - C)$

$\frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C$

$\tan A + \tan B = -\tan C(1 - \tan A\tan B)$

$\therefore \quad \tan A + \tan B + \tan C = \tan A\tan B\tan C$

2. $L.H.S. = 1 - (1 - 2\sin^2 A) + 2\sin\frac{6A+4A}{2}\sin\frac{6A-4A}{2} = 2\sin^2 A + 2\sin 5A\sin A$

$= 2\sin A(\sin A + \sin 5A) = 2\sin A\left(2\sin\frac{5A+A}{2}\cos\frac{5A-A}{2}\right) = R.H.S.$

3. $L.H.S. = (1 - \cos 2A) + (\cos 4A - \cos 6A) = 2\sin^2 A + 2\sin\frac{6A+4A}{2}\sin\frac{6A-4A}{2}$

$= 2\sin^2 A + 2\sin 5A\sin A = 2\sin A(\sin A + \sin 5A) = 2\sin A\left(2\sin\frac{5A+A}{2}\cos\frac{5A-A}{2}\right) = R.H.S.$

4. $L.H.S. = \sin A \left(2\sin \frac{5A+3A}{2}\cos\frac{5A-3A}{2}\right) = 2\sin A\sin 4A\cos A$

$= \cos A(2\sin A\sin 4A) = \cos A[\cos(4A - A) - \cos (4A + A)] = R.H.S.$

5. **Method 1**

$L.H.S. = \tan A + \frac{\tan A+\tan 120°}{1-\tan A\tan 120°} + \frac{\tan A+\tan 240°}{1-\tan A\tan 240°} = \tan A + \frac{\tan A-\sqrt{3}}{1-\tan A(-\sqrt{3})} + \frac{\tan A+\sqrt{3}}{1-\tan A(\sqrt{3})}$

$$= \frac{\tan A(1+\sqrt{3}\tan A)(1-\sqrt{3}\tan A)+(1-\sqrt{3}\tan A)(\tan A-\sqrt{3})+(1+\sqrt{3}\tan A)(\tan A+\sqrt{3})}{(1+\sqrt{3}\tan A)(1-\sqrt{3}\tan A)}$$

$$= \frac{\tan A(1-3\tan^2 A)+(\tan A-\sqrt{3}\tan^2 A-\sqrt{3}+3\tan A)+(\tan A+\sqrt{3}\tan^2 A+\sqrt{3}+3\tan A)}{1-3\tan^2 A} = 3\,\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$

$$\mathrm{R.H.S} = 3\tan(2A + A) = 3\,\frac{\tan 2A+\tan A}{1-\tan 2A\tan A} = 3\,\frac{\frac{2\tan A}{1-\tan^2 A}+\tan A}{1-\frac{2\tan A}{1-\tan^2 A}\tan A} = 3\,\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

Method 2

$$\tan 3x = \tan(2x + x) = \frac{\tan 2x+\tan x}{1-\tan 2x\tan x} = \frac{\frac{2\tan x}{1-\tan^2 x}+\tan x}{1-\frac{2\tan x}{1-\tan^2 x}\tan x} = \frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$$

Rearrange, $(1 - 3\tan^2 x)\tan 3x = 3\tan x - \tan^3 x$

Or $\tan^3 x - (3\tan 3x)\tan^2 x - 3\tan x + \tan 3x = 0$

Construct the equation:

$$f(t) = t^3 - (3\tan 3x)t^2 - 3t + \tan 3x = 0$$

Observe that tan 3A = tan 3(A + 120°) = tan 3(A + 240°)

Then tan A , tan (A + 120°) , tan (A + 240°) are roots of f(t) = 0.

Since f(t) = 0 is a cubic equation and has three roots,

Sum of roots $= \tan A + \tan(A + 120°) + \tan(A + 240°) = -\text{coeff. of } t^2 - \text{term} = 3\tan 3x$

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